

GK-DIMENSION OF THE LIE ALGEBRA OF GENERIC 2×2 MATRICES

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ABSTRACT. Recently Machado and Koshlukov have computed the Gelfand-Kirillov dimension of the relatively free algebra $F_m = F_m(\text{var}(sl_2(K)))$ of rank m in the variety of algebras generated by the three-dimensional simple Lie algebra $sl_2(K)$ over an infinite field K of characteristic different from 2. They have shown that $\text{GKdim}(F_m) = 3(m - 1)$. The algebra F_m is isomorphic to the Lie algebra generated by m generic 2×2 matrices. Now we give a new proof for $\text{GKdim}(F_m)$ using classical results of Procesi and Razmyslov combined with the observation that the commutator ideal of F_m is a module of the center of the associative algebra generated by m generic traceless 2×2 matrices.

1. INTRODUCTION

Let R be a (not necessarily associative) algebra generated by m elements r_1, \dots, r_m over a field K and let V_n be the vector subspace of R spanned by all products $r_{i_1} \cdots r_{i_k}$, $k \leq n$. The growth function of R with respect to the given system of generators is

$$g_R(n) = \dim(V_n), \quad n \geq 0.$$

The Gelfand-Kirillov dimension of R is defined as

$$\text{GKdim}(R) = \limsup_{n \rightarrow \infty} \log_n(g_R(n)).$$

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It does not depend on the choice of the generators of R . See the book [9] for a background on GKdim. If the algebra R is graded,

$$R = \bigoplus_{n \geq 0} R^{(n)},$$

where $R^{(n)}$ is the homogeneous component of degree n of R , then the Hilbert series of R is the formal power series

$$H(R, t) = \sum_{n \geq 0} \dim(R^{(n)}) t^n.$$

If R is generated by its homogeneous elements of first degree, then its growth function is

$$g_R(n) = \sum_{l=0}^n \dim(R^{(l)}).$$

In the general case, if R is a graded algebra generated by a finite system of (homogeneous) elements of arbitrary degree, its Gelfand-Kirillov dimension can be expressed using again its Hilbert series as

$$\text{GKdim}(R) = \limsup_{n \rightarrow \infty} \log_n \left(\sum_{l=0}^n \dim(R^{(l)}) \right).$$

When studying varieties of K -algebras \mathfrak{V} , all information for the m -generated algebras in \mathfrak{V} is carried by the relatively free algebra $F_m(\mathfrak{V})$ of rank m in \mathfrak{V} . When the base field K is of characteristic 0, a lot is known for the Gelfand-Kirillov dimension of relatively free associative algebras, see the book [9], the survey article [4], or the paper [11]. In particular, $\text{GKdim}(F_m(\mathfrak{V}))$ is an integer for all proper varieties of associative algebras. Almost nothing is known for relatively free Lie algebras. Using the bases of free nilpotent-by-abelian Lie algebras given by Shmelkin [17], it is easy to see that

$$\text{GKdim}(F_m(\mathfrak{N}_c \mathfrak{A})) = \text{GKdim}(L_m / (L'_m)^{c+1}) = mc,$$

where $m > 1$ and L_m is the free m -generated Lie algebra. Together with free nilpotent Lie algebras where the Gelfand-Kirillov dimension is equal to 0, these are the only free polynilpotent Lie algebras of finite Gelfand-Kirillov dimension, see Petrogradsky [12].

Recently Machado and Koshlukov [11] have computed the Gelfand-Kirillov dimension of the relatively free algebra $F_m = F_m(\text{var}(sl_2(K)))$ of rank $m > 2$ in the variety of algebras generated by the three-dimensional simple Lie algebra $sl_2(K)$ over an infinite field K of characteristic different from 2. They have shown that $\text{GKdim}(F_m) = 3(m-1)$. Their proof is based on a careful analysis of the explicit expression of

the Hilbert series of F_m obtained by Drensky [3]. The case $m = 2$ was handled before by Bahturin [2] who showed that $\text{GKdim}(F_2) = 3$. The algebra F_m is isomorphic to the Lie algebra generated by m generic traceless 2×2 matrices. The purpose of our paper is to give a new proof for $\text{GKdim}(F_m)$ using classical results of Procesi [13, 14] on Gelfand-Kirillov dimension of the algebra of generic matrices and Razmyslov [16] on the weak polynomial identities of matrices, combined with the observation that the commutator ideal of F_m is a module over the center of the associative algebra generated by m generic traceless 2×2 matrices. We believe that the present approach is more adequate for generalizations for other finite dimensional simple Lie algebras than the approach in [11].

2. THE PROOF

The following statement and its corollary are folklore known. We include the proof for self-completeness of the exposition and also because we were not able to find an explicit reference.

Lemma 1. *Let R be a finitely generated graded algebra with Hilbert series of the form*

$$H(R, t) = h(t) \prod_{i=1}^s \frac{1}{(1 - t^{d_i})},$$

where $h(t) \in \mathbb{C}[t]$ is a polynomial and the d_i 's are positive integers. Then the Gelfand-Kirillov dimension of R is equal to the multiplicity of 1 as a pole of $H(R, t)$.

Proof. It is sufficient to consider the case when R is not finite dimensional and hence its Hilbert series has a nontrivial denominator. Let d be the least common multiple of the degrees d_i . Then

$$\begin{aligned} H(R, t) &= \sum_{n \geq 0} a_n t^n = f(t) + \sum_{p=1}^k \sum_{q=0}^{d-1} \frac{\alpha_{pq}}{(1 - \omega_q t)^p} \\ &= f(t) + \sum_{n \geq 0} \left(\sum_{p=1}^k \binom{n+p-1}{p-1} \sum_{q=0}^{d-1} \alpha_{pq} \omega_q^n \right) t^n, \end{aligned}$$

where $f(t) \in \mathbb{C}[t]$, $\alpha_{pq} \in \mathbb{C}$, $\omega_0 = 1, \omega_1, \dots, \omega_{d-1}$ are the d -th roots of 1, and at least one of the coefficients α_{kq} is different from zero. Since $\omega_q^d = 1$, the sequences

$$\beta_{pn} = \sum_{q=0}^{d-1} \alpha_{pq} \omega_q^n, \quad p = 1, \dots, k,$$

are periodic with period d and for n large enough the coefficients a_n of the Hilbert series $H(R, t)$ are bounded by polynomials of degree $k - 1$ in n . Hence the sequence

$$\sum_{l=0}^n a_l = \sum_{l=0}^n \dim(R^{(l)})$$

needed for the definition of the Gelfand-Kirillov dimension of R is bounded by a polynomial of degree k in n and

$$\text{GKdim}(R) \leq k.$$

The asymptotics of the coefficients a_n of

$$H(R, t) = f(t) + \sum_{n \geq 0} \left(\sum_{p=1}^k \binom{n+p-1}{p-1} \beta_{pn} \right) t^n,$$

is determined by β_{kn} . Since a_n are positive integers, we derive that the periodic sequence β_{kn} , $n = 0, 1, 2, \dots$, consists of nonnegative reals and at least one of them is positive. Since $\omega_q^d = 1$, if $\omega_q \neq 1$, then $1 + \omega_q + \omega_q^2 + \dots + \omega_q^{d-1} = 0$. Hence

$$\begin{aligned} 0 &< \sum_{l=0}^{d-1} \beta_{k, dn+l} = \sum_{l=0}^{d-1} \sum_{q=0}^{d-1} \alpha_{kq} \omega_q^{dn+l} \\ &= \sum_{q=0}^{d-1} \alpha_{kq} \sum_{l=0}^{d-1} \omega_q^l = d\alpha_{k0}. \end{aligned}$$

Therefore $\alpha_{k0} > 0$. We consider the partial sum $p_{dn} = a_0 + a_1 + \dots + a_{dn}$ of the coefficients of the Hilbert series $H(R, t)$. Its asymptotics is determined by

$$\begin{aligned} \tilde{p}_{dn} &= \sum_{c=0}^{dn} \binom{c+k-1}{k-1} \beta_{kc} \approx \frac{1}{(k-1)!} \sum_{c=0}^{dn} c^{k-1} \beta_{kc} \\ &\approx \frac{1}{(k-1)!} \sum_{e=0}^n (ed)^{k-1} \sum_{l=0}^{d-1} \beta_{k, ed+l} = \frac{d\alpha_{k0}}{(k-1)!} \sum_{e=0}^n (ed)^{k-1} \end{aligned}$$

and this is a polynomial of degree k in n . Hence

$$\begin{aligned} \text{GKdim}(R) &= \limsup_{n \rightarrow \infty} \log_n \left(\sum_{l=0}^n a_l \right) \geq \limsup_{n \rightarrow \infty} \log_n \left(\sum_{c=0}^{dn} a_c \right) \\ &= \limsup_{n \rightarrow \infty} \log_{dn} (p_{dn}) = \limsup_{n \rightarrow \infty} \log_{dn} (\tilde{p}_{dn}) = k \end{aligned}$$

which, together with the opposite inequality $\text{GKdim}(R) \leq k$, completes the proof. \square

Corollary 2. *Let R be a finitely generated graded algebra and let C be a finitely generated graded subalgebra of the center of R such that R is a finitely generated C -module. Then the Gelfand-Kirillov dimension of R is equal to the multiplicity of 1 as a pole of $H(R, t)$.*

Proof. By the Hilbert-Serre theorem (see e.g., [1]), the Hilbert series of any finitely generated graded module M over a finitely generated graded commutative algebra C is of the form

$$H(M, t) = h(t) \prod_{i=1}^k \frac{1}{(1 - t^{d_i})}, \quad h(t) \in \mathbb{C}[t], d_i > 0.$$

Hence the proof follows immediately from Lemma 1. \square

In the sequel we assume that the base field K is of characteristic 0. Let

$$\Omega_{km} = K[Y_{km}] = K[y_{pq}^{(i)} \mid p, q = 1, \dots, k, i = 1, \dots, m]$$

be the algebra of polynomials in $k^2 m$ commuting variables and let

$$y_i = (y_{pq}^{(i)}), \quad i = 1, \dots, m,$$

be m generic $k \times k$ matrices. We consider the following algebras:

R_{km} – the generic matrix algebra. This is the subalgebra generated by y_1, \dots, y_m of the associative $k \times k$ matrix algebra $M_k(\Omega_{km})$ with entries from Ω_{km} .

C_{km} – the pure trace algebra. This is the subalgebra of Ω_{km} generated by the traces of the products, $\text{tr}(y_{i_1} \cdots y_{i_l})$. We embed C_{km} in $M_k(\Omega_{km})$ by $f(Y_{km}) \rightarrow f(Y_{km})I_k$, where I_k is the identity matrix.

T_{km} – the mixed trace algebra. This is the subalgebra of $M_k(\Omega_{km})$ generated by R_{km} and C_{km} .

For a background on generic matrices see e.g., [14] or [7]. Below we summarize the results we need.

Proposition 3. *Let $k, m \geq 2$. Then:*

- (i) *The mixed trace algebra T_{km} has no zero divisors;*
- (ii) *The pure trace algebra C_{km} coincides with the center of T_{km} . It is finitely generated and T_{km} is a finitely generated C_{km} -module;*
- (iii) [8, 13]

$$\text{GKdim}(T_{km}) = \text{GKdim}(C_{km}) = \text{GKdim}(R_{km}) = k^2(m-1) + 1.$$

Further, we consider the generic traceless $k \times k$ matrices

$$z_i = (z_{pq}^{(i)}) = y_i - \frac{1}{k} \text{tr}(y_i) I_k, \quad i = 1, \dots, m,$$

and the subalgebra W_{km} of T_{km} generated by z_1, \dots, z_m , the subalgebra $C_{km}^{(0)}$ of C_{km} generated by the traces of the products, $\text{tr}(z_{i_1} \cdots z_{i_l})$, and

the subalgebra $T_{km}^{(0)}$ of T_{km} generated by W_{km} and $C_{km}^{(0)}$. Finally, let L_{km} be the Lie subalgebra of W_{km} generated by z_1, \dots, z_m .

Proposition 4. *Let $k, m \geq 2$. Then*

(i) (Procesi [15])

$$T_{km} \cong K[\operatorname{tr}(y_1), \dots, \operatorname{tr}(y_m)] \otimes_K T_{km}^{(0)},$$

$$C_{km} \cong K[\operatorname{tr}(y_1), \dots, \operatorname{tr}(y_m)] \otimes_K C_{km}^{(0)};$$

(ii) (Razmyslov [16])

$$W_{km} \cong K\langle x_1, \dots, x_m \rangle / \operatorname{Id}(M_k(K), \operatorname{sl}_k(K))$$

where $\operatorname{Id}(M_k(K), \operatorname{sl}_k(K))$ is the ideal of all weak polynomial identities in m variables for the pair $(M_k(K), \operatorname{sl}_k(K))$, i.e., the polynomials in the free associative algebra $K\langle x_1, \dots, x_m \rangle$ which vanish when evaluated on $\operatorname{sl}_k(K)$ considered as a subspace in $M_k(K)$.

(iii) (Razmyslov [16]) *The Lie algebra L_{km} is isomorphic to the relatively free algebra $F_m(\operatorname{var}(\operatorname{sl}_k)(K))$ in the variety of Lie algebras generated by $\operatorname{sl}_k(K)$.*

Corollary 5. *For $k, m \geq 2$*

$$\operatorname{GKdim}(T_{km}^{(0)}) = \operatorname{GKdim}(C_{km}^{(0)}) = (k^2 - 1)(m - 1).$$

Proof. The algebras T_{km} and C_{km} satisfy the conditions of Corollary 2. Hence the multiplicity of 1 as a pole of the Hilbert series of T_{km} and C_{km} is equal to their Gelfand-Kirillov dimension $k^2(m - 1) + 1$ (see Proposition 3 (iii)). Proposition 4 (i) gives that

$$H(T_{km}, t) = H(K[\operatorname{tr}(y_1), \dots, \operatorname{tr}(y_m)], t) H(T_{km}^{(0)}, t) = \frac{1}{(1 - t)^m} H(T_{km}^{(0)}, t),$$

$$H(C_{km}, t) = \frac{1}{(1 - t)^m} H(C_{km}^{(0)}, t).$$

Hence the multiplicity of 1 as a pole of $H(T_{km}^{(0)}, t)$ and $H(C_{km}^{(0)}, t)$ is equal to $(k^2(m - 1) + 1) - m = (k^2 - 1)(m - 1)$. Both algebras $T_{km}^{(0)}$ and $C_{km}^{(0)}$ are finitely generated and graded. Hence the proof follows from Corollary 2. \square

Now we shall summarize the information for 2×2 generic matrices.

Proposition 6. *Let $k = 2$ and $m \geq 2$. Then:*

(i) (Sibirskii [18]) *The trace polynomials*

$$\operatorname{tr}(y_i), \quad i = 1, \dots, m, \quad \operatorname{tr}(y_i y_j), \quad 1 \leq i \leq j \leq m,$$

$$\operatorname{tr}(y_{i_1} y_{i_2} y_{i_3}), \quad 1 \leq i_1 < i_2 < i_3 \leq m,$$

form a minimal system of generators of C_{2m} .

(ii) (Procesi [15]) *The algebras $T_{2m}^{(0)}$ and W_{2m} coincide. The algebra $C_{2m}^{(0)}$ is generated by*

$$\text{tr}(z_i z_j), 1 \leq i \leq j \leq m, \quad \text{tr}(z_{i_1} z_{i_2} z_{i_3}), 1 \leq i_1 < i_2 < i_3 \leq m,$$

which belong to W_{2m} .

(iii) (Drensky [5]) *The algebra $C_{2m}^{(0)}$ is generated by*

$$z_i^2, i = 1, \dots, m, \quad z_i z_j + z_j z_i, 1 \leq i \leq j \leq m, \\ s_3(z_{i_1}, z_{i_2}, z_{i_3}), 1 \leq i_1 < i_2 < i_3 \leq m,$$

where

$$s_3(x_1, x_2, x_3) = \sum_{\sigma \in S_3} \text{sign}(\sigma) x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}$$

is the standard polynomial of degree 3.

Proof. We shall present the proof of (ii) and (iii) as a consequence of (i). Clearly $C_{2m}^{(0)}$ is generated by $\text{tr}(z_i z_j)$, $1 \leq i \leq j \leq m$, and $\text{tr}(z_{i_1} z_{i_2} z_{i_3})$, $1 \leq i_1 < i_2 < i_3 \leq m$. Now the proof of (ii) and (iii) follows immediately from the equalities in $T_{2m}^{(0)}$

$$\text{tr}(z_1^2) = 2z_1^2, \quad \text{tr}(z_1 z_2) = z_1 z_2 + z_2 z_1, \\ \text{tr}(z_1 z_2 z_3) = \frac{1}{3} s_3(z_1, z_2, z_3)$$

which may be checked by direct verification. \square

Lemma 7. *The commutator ideal L'_{2m} is a $C_{2m}^{(0)}$ -module.*

Proof. The following equalities which can be verified directly hold in W_{2m} :

$$[z_1, z_2] z_3^2 = \frac{1}{4} ([z_1, z_2, z_3, z_3] - [[z_1, z_3], [z_2, z_3]]), \\ [z_1, z_2] (z_3 z_4 + z_4 z_3) = \frac{1}{4} ([z_1, z_2, z_3, z_4] + [z_1, z_2, z_4, z_3] \\ - [[z_1, z_3], [z_2, z_4]] - [[z_1, z_4], [z_2, z_3]]), \\ z_4 s_3(z_1, z_2, z_3) = \frac{3}{8} \sum_{\sigma \in S_3} \text{sign}(\sigma) [z_4, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}].$$

The elements of the commutator ideal are linear combinations of (left normed) commutators $u_i = [z_{i_1}, z_{i_2}, \dots, z_{i_n}]$. If v is a generator of $C_{2m}^{(0)}$, then

$$u_i v = [z_{i_1}, z_{i_2}, \dots, z_{i_n}] v = [[z_{i_1}, z_{i_2}] v, \dots, z_{i_n}]$$

and the above equalities guarantee that $u_i v$ is a linear combination of commutators, i.e., belongs to L'_{2m} again. Hence $L'_{2m} C_{2m}^{(0)} \subset L'_{2m}$. \square

Remark 8. It is known that W_{2m} is a $C_{2m}^{(0)}$ -module generated by 1, z_i , $i = 1, \dots, m$, and $[z_i, z_j]$, $1 \leq i < j \leq m$. Using the equality

$$[z_1, z_2, z_3] = 2(z_1(z_2z_3 + z_3z_2) - z_2(z_1z_3 + z_3z_1)),$$

as in the proof of Lemma 7 we can show that L'_{2m} is a $C_{2m}^{(0)}$ -module generated by all commutators $[z_i, z_j]$ and $[z_{i_1}, z_{i_2}, z_{i_3}]$. For $m = 2$, the commutator ideal L'_{22} is a free $C_{22}^{(0)}$ -module generated by $[z_1, z_2]$, $[z_1, z_2, z_1]$, $[z_1, z_2, z_2]$, see [6].

The proof of the following theorem established in [11] is the main result of our paper.

Theorem 9. *Let K be a field of characteristic 0 and let L_{2m} be the Lie algebra generated by m generic traceless 2×2 matrices, $m \geq 2$. Then*

$$\text{GKdim}(L_{2m}) = \text{GKdim}(F_m(\text{var}(sl_2(K)))) = 3(m-1).$$

Proof. Let

$$H(C_{2m}^{(0)}, t) = \sum_{n \geq 0} c_n t^n, \quad H(L_{2m}, t) = \sum_{n \geq 1} l_n t^n, \quad H(W_{2m}, t) = \sum_{n \geq 1} w_n t^n$$

be the Hilbert series of $C_{2m}^{(0)}$, L_{2m} , and W_{2m} , respectively. Since the algebra L_{2m} is finitely generated, its Gelfand-Kirillov dimension is

$$\text{GKdim}(L_{2m}) = \limsup_{n \rightarrow \infty} \log_n \left(\sum_{k=1}^n l_k \right).$$

The algebra W_{2m} has no zero divisors and hence $[z_1, z_2]C_{2m}^{(0)} \subset L'_{2m} \subset L_{2m}$ is a free $C_{2m}^{(0)}$ -module. Therefore

$$\sum_{k=0}^{n-2} c_k \leq \sum_{k=1}^n l_k \leq \sum_{k=0}^n w_k,$$

which implies that

$$3(m-1) = \text{GKdim}(C_{2m}^{(0)}) \leq \text{GKdim}(L_{2m}) \leq \text{GKdim}(W_{2m}) = 3(m-1).$$

□

Remark 10. As in [11], the formula for the Gelfand-Kirillov dimension of $F_m(\text{var}(sl_2(K)))$ obtained in characteristic 0 holds also for any infinite field K of characteristic different from 2.

Remark 11. In characteristic 2, the algebra $sl_2(K)$ is nilpotent of class 2 and hence $F_m(\text{var}(sl_2(K)))$ is isomorphic to the free nilpotent of class 2 Lie algebra $F_m(\mathfrak{N}_2)$ which is finite dimensional. Therefore

$\text{GKdim}(F_m(\text{var}(sl_2(K)))) = 0$. When K is an infinite field of characteristic 2, a much more interesting object is the relatively free algebra $F_m(\text{var}(M_2(K)^{(-)}))$ of the variety generated by the 2×2 matrix algebra $M_2(K)$ considered as a Lie algebra. Vaughan-Lee [19] showed that the algebra $M_2(K)^{(-)}$ does not have a finite basis of its polynomial identities. (It is easy to see that the four-dimensional Lie algebra constructed in [19] is isomorphic to $M_2(K)^{(-)}$.) The algebra $M_2(K)^{(-)}$ satisfies the center-by-metabelian polynomial identity

$$[[[x_1, x_2], [x_3, x_4]], x_5] = 0.$$

It is well known that the free center-by-metabelian Lie algebra $F_m([\mathfrak{A}^2, \mathfrak{E}])$ over any field K is spanned by

$$[x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_n}], \quad [[x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_n}], [x_{i_{n+1}}, x_{i_{n+2}}]],$$

where $i_1 > i_2 \leq i_3 \leq \dots \leq i_n$ and the commutators are left normed, e.g., $[x_1, x_2, x_3] = [[x_1, x_2], x_3]$. (A basis of $F_m([\mathfrak{A}^2, \mathfrak{E}])$ is given by Kuzmin [10].) Since the commutators $[x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_n}]$ form a basis of the free metabelian Lie algebra $F_m(\mathfrak{A}^2)$ and are linearly independent in $F_m(\text{var}(M_2(K)^{(-)}))$, we obtain immediately that

$$\text{GKdim}(F_m(\text{var}(M_2(K)^{(-)}))) = m, \quad m > 1.$$

In characteristic 2 there is another three-dimensional simple Lie algebra which is an analogue of the Lie algebra of the three-dimensional real vector space with the vector multiplication. It is interesting to see whether this algebra has a finite basis of its polynomial identities (probably not) and, when the field is infinite, to compute the Gelfand-Kirillov dimension of the corresponding relatively free algebras.

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